Linear programming I

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- Linear programming (LP) or linear optimization is a set of optimization algorithms.
- Short history: began in 1947 when George Dantzig devised the simplex method.

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- Goal: Optimize **linear** objective functions, subject to (s.t.) a set of **linear** constraints (equality or inequality).
- Widely used in various industries for production planning, investment assignment, transportation problem, traffic control, etc. The LP class is usually offered by Industrial Engineering department.
- Many applications in statistics: regularized regression with L₁ penalty (e.g., LASSO), quantile regression, support vector machine, etc.

Consider a hypothetical company that manufactures two products A and B.

- 1. Each item of product A requires 1 hour of labor and 2 units of materials, and yields a profit of 1 dollar.
- 2. Each item of product B requires 2 hours of labor and 1 unit of materials, and yields a profit of 1 dollar.
- 3. The company has 100 hours of labor and 100 units of materials available.

Question: what's the company's optimal strategy for production?

The problem can be formulated as following optimization problem. Denote the amount of production for product A and B by x_1 and x_2 :

 $max \ z = x_1 + x_2,$ s.t. $x_1 + 2x_2 \le 100$ $2x_1 + x_2 \le 100$ $x_1, x_2 \ge 0$ The small 2-variable problem can be represented in a 2D graph.

The isoprofit line (any point on the line generate the same profit) is perpendicular to the gradient of objective function. The problem can be solved by sliding the isoprofit line.

Optimal solution might not exist (e.g., the objective function is unbounded in the solution space).

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But if the optimal solution exists:

- Interior point: No.
- Corner point: Yes. The corner points are often referred to as "extreme points".
- Edge point: Yes, only when the edge is parallel to the isoprofit line. In this case, every point on that edge has the same objective function value, including the corner point of that edge.
- Key result: If an LP has an optimal solution, it must have a extreme point optimal solution (can be rigorously proved using the General Representation Theorem). This greatly reduce our search of optimal solutions: only need to check a finite number of extreme points.

The inequality constrains can be converted into equality constrains by adding non-negative "**slack**" variables. For example,

```
x_1 + 2x_2 \le 100 \iff x_1 + 2x_2 + x_3 = 100, x_3 \ge 0.
```

Here x_3 is called a "slack" variable.

The original problem can be expressed as the following "slack form":

```
max \ z = x_1 + x_2,
s.t. x_1 + 2x_2 + x_3 = 100
2x_1 + x_2 + x_4 = 100
x_1, x_2, x_3, x_4 \ge 0
```

This augmentation is necessary for the **simplex** algorithm.

The augmented standard form of LP problem, expressed in matrix notation is:

```
max \ z = c\mathbf{x}s.t. \ A\mathbf{x} = b\mathbf{x} \ge 0
```

It is required that RHS $b \ge 0$. Here x is augmented, e.g., including the original and slack variables. For our simple example, we have:

```
\mathbf{x} = [x_1, x_2, x_3, x_4]^T
c = [1, 1, 0, 0]
A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}
b = [100, 100]^T
```

We know the problem has an optimal solution at the extreme point if the optimal solution exists \implies we can try all extreme points and find the best one.

Assume there are *m* constraints an *n* unknowns (so *A* is of dimension $m \times n$). In our case m = 2 and n = 4.

- It is required that m < n (why?)
- Assume rank(A) = m, e.g., rows of A are independent (no redundant constraints).
- An extreme point is the intersection of *n* linearly independent hyperplanes. The constraints $A\mathbf{x} = b$ provide *m* such hyperplanes. How about the remaining n m of them?

Solution: first set n - m of the unknowns to zero. Essentially this gets rid of n - m variables and leaves us with m unknowns and m equations, which can be solved.

Terminology: The n - m variables set to zero are called *nonbasic variables* (NBV) and denoted by \mathbf{x}_N . The rest m variables are *basic variables* (BV), denoted by \mathbf{x}_B .

Let *B* be the columns of *A* that are associated with the basic variables, and *N* be the columns associated with nonbasic variables (*B* is square and has full rank):

$$A\mathbf{x} = b \Longrightarrow [B, N] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = b \Longrightarrow \mathbf{x}_B = B^{-1}b$$

So the point $\mathbf{x}_N = 0$, $\mathbf{x}_B = B^{-1}b$ is an extreme point.

Bad news: there are too many of them! With 20 variables and 10 constraints (which is considered a very small problem), there are $\binom{20}{10} = 184,756$ extreme points.

This grows quickly. With 50 variables and 20 constrains (still a small problem), there are 4.7×10^{13} extreme points.

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Given the LP problem

 $max \ z = c\mathbf{x}, \quad s.t. \ A\mathbf{x} = b, \quad \mathbf{x} \ge 0$

First characterize the solution at an extreme point:

$$B\mathbf{x}_B + N\mathbf{x}_N = b \Longrightarrow \mathbf{x}_B = B^{-1}b - B^{-1}N\mathbf{x}_N \Longrightarrow \mathbf{x}_B = B^{-1}b - \sum_{j \in R} B^{-1}\mathbf{a}_j x_j$$

Here *R* is the set of nonbasic variables, and \mathbf{a}_i 's are columns of *A*.

Substitute this expression to the objective function. Let c_B and c_N be the coefficients in the objective function for x_B and x_N :

$$z = c\mathbf{x} = c_B \mathbf{x}_B + c_N \mathbf{x}_N$$

= $c_B \left[B^{-1}b - \sum_{j \in R} B^{-1} \mathbf{a}_j x_j \right] + \sum_{j \in R} c_j x_j$
= $c_B B^{-1}b - \sum_{j \in R} (c_B B^{-1} \mathbf{a}_j - c_j) x_j$

The current extreme point is optimal if $c_B B^{-1} a_j - c_j \ge 0$ for $\forall j$ (why?).

Iterative method:

- No need to enumerate all extreme points.
- Never go to extreme point which yield smaller objective function, e.g., the objective function always increases during iterations.

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Consider our simple example with:

$$c = [1, 1, 0, 0], A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, b = [100, 100]^T$$

First assume we choose x_1 and x_3 as basic variables, then

$$c_B = [1,0], c_N = [1,0], B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, R = \{2,4\}$$

Solving $x_B = B^{-1}b = [50, 50]^T$. Are we optimal?

Check $c_B B^{-1} \mathbf{a}_j - c_j \equiv w_j$ for $j \in \mathbf{R}$.

$$w_{2} = \begin{bmatrix} 1, 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 = -0.5$$
$$w_{4} = \begin{bmatrix} 1, 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 0.5$$

- $w_2 < 0$ so we know the solution is not optimal \implies increasing x_2 will increase the objective function z.
- Since we will increase x_2 , it will no longer be a nonbasic variable (will not be zero). It is referred to as **entering basic variable**.
- But when we increase x_2 , it will change the values of other variables. How much can we increase x_2 ?

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Now go back to look at $x_B = B^{-1}b - \sum_{j \in R} B^{-1}\mathbf{a}_j x_j$. To simplify the notations, define $y_j = B^{-1}\mathbf{a}_j$ for $j \in R$, and $\overline{b} = B^{-1}b$.

$$y_{2} = B^{-1}a_{2} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$
$$y_{4} = B^{-1}a_{4} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$
$$\bar{b} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

Plug these back to the expessions for x_B :

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} x_2 - \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} x_4$$

Holding $x_4 = 0$ and increasing x_2 , x_1 or x_3 will eventually become negative.

How much can we increase $x_2 \implies$ increasing x_2 , which basic variable (x_1 and x_3) will hit zero first?

The basic variable that hits zero first is called the **leaving basic variable**. Given index of entering basic variable k, we find index for the leaving basic variable, denoted by l, as following:

$$l = \operatorname{argmin}_{i} \left\{ \frac{\bar{b}_i}{y_{ki}}, 1 \le i \le m, y_{ki} > 0 \right\}$$

For our example, k = 2, $\bar{b}_1/y_{21} = 50/0.5 = 100$, $\bar{b}_2/y_{22} = 50/1.5 = 33.3$. So the **second** basic variable is leaving, which is x_3 .

Be careful of the indexing here (especially in programming). It is the second of the current basic variable that is leaving, not x_2 !

Next iteration, with x_1 and x_2 as basic variables. We will have

$$c_B = [1, 1], c_N = [0, 0], B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = \{3, 4\}$$

We will get $x_B = [33.3, 33.3]^T$, and

$$w_{3} = \begin{bmatrix} 1, 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = 0.33$$
$$w_{4} = \begin{bmatrix} 1, 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 0.33$$

Both w's are positive so we are at optimal solution now!

- It only considers extreme points.
- It moves as far as it can until the movement is blocked by another constraint.

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• It's greedy: find the edge that optimize the objective function the fastest and move along that edge.

To make it formal, the **steps for Simplex method** are:

- 1. Randomly choose a set of basic variables (often use the slack variables).
- 2. Find entering basic variable. Define the set of nonbasic variable to be *R*. For each $j \in R$, compute $C_B B^{-1} \mathbf{a}_j c_j \equiv \omega_j$. If all $\omega_j \ge 0$, the current solution is optimal. Otherwise find $k = \operatorname{argmin}_{j \in R} \omega_j$, the *k*-th nonbasic variable will be EBV.
- 3. Find leaving basic variable. Obtain $l = argmin_i \left\{ \frac{\bar{b}_i}{y_{ki}}, 1 \le i \le m, y_{ki} > 0 \right\}$. The *l*-th current basic variable will be LBV.
- 4. Iterate steps 2 and 3.

The LP problem: $max \ z = cx$, $s.t. \ Ax = b$, $x \ge 0$. With BV and NBV and corresponding matrices, we can reorganize them as:

$$z = c_B B^{-1} b - (c_B B^{-1} N - c_N) \mathbf{x}_N.$$
$$\mathbf{x}_B = B^{-1} b - B^{-1} N \mathbf{x}_N.$$

This is to express the objective function and BVs in terms of NBVs and constants. We do this because:

- Expressing *z* in terms of NBVs and constants so that we will know whether the current solution is optimal, and how to choose EBV.
- Expressing BVs in terms of NBVs and constants in order to choose LBV.

This can be organized into following tableau form:

	Z	x_B	x_N	RHS
Z.	1	0	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$
x_B	0		$B^{-1}N$	$B^{-1}b$

The LP problem is:

 $max \quad 50x_1 + 30x_2 + 40x_3$ s.t. $2x_1 + 3x_2 + 5x_3 \le 100$ $5x_1 + 2x_2 + 4x_3 \le 80$ $x_1, x_2, x_3 \ge 0$

Step 1: Add slack variables (x_4, x_5) to get equality constraints.

Step 2: Choose initial basic variables - the easiest is to choose the slack variables as BV, then put into tableau (verify):

	Z	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
Z.	1	-50	-30	-40	0	0	0
<i>x</i> ₄	0	2	3	5	1	0	100
<i>x</i> ₅	0	5	2	4	0	1	80

Choose x_1 to be EBV (why?).

Step 3: Now to find LBV, need to compute ratios of the RHS column to the EBV column, get $x_4 : 100/2$, $x_5 : 80/5$. We need to take the BV with minimum of these ratios, e.g., x_5 , as the LBV.

Step 4: Update the tableau: replace x_5 by x_1 in BV. We must update the table so that x_1 has coefficient of 1 in row 2 and coefficient 0 in every other row. This step is call **pivoting**.

To do so, the new row 3 =old row 3 divided by 5:

	Z	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
Z.	1						
<i>x</i> ₄	0						
<i>x</i> ₁	0	1	2/5	4/5	0	1/5	16

To eliminate the coefficient of x_1 in row 1 and row 2, we perform row operations:

- new row $1 = \text{old row } 1 + 50 \times \text{new row } 3$.
- new row 2 = old row 2 $2 \times$ new row 3.

	Z	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
Z	1	0	-10	0	0	10	800
<i>x</i> ₄	0	0	11/5	17/5	1	-2/5	68
<i>x</i> ₁	0	1	2/5	4/5	0	1/5	16

Step 5: Now x_2 will be EBV (because it has negative coefficient). By calculating ratios, x_4 is LBV (verify!). Another step of pivoting gives:

	z	x_1	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
Z	1	0	0	$15\frac{5}{11}$	$4\frac{6}{11}$	$8\frac{2}{11}$	1109 $\frac{1}{11}$
<i>x</i> ₂	0	0	1	17/11	5/11	-2/11	$30\frac{10}{11}$
<i>x</i> ₁	0	1	0	2/11	-2/11	3/11	$3\frac{7}{11}$

Now all coefficients for NBV are positive, meaning we have reached the optimal solution.

The RHS column gives the optimal objective function value, as well as the values of BVs at the optimal solution.

Handling > constraints

So far we assume all constraints are \leq with non-negative RHS. What is some constraints are \geq ? For example, the following LP problem:

 $max - x_1 - 3x_2$ s.t. $x_1 + 2x_2 \ge 1$ $5x_1 + x_2 \le 10$ $x_1, x_2 \ge 0$ — 20/39 -

For the \geq constraint we can subtract a slack variable ("surplus variable"):

$$max - x_1 - 3x_2$$

s.t. $x_1 + 2x_2 - x_3 = 1$
 $5x_1 + x_2 + x_4 = 10$
 $x_1, x_2, x_3, x_4 \ge 0$

The problem now is that we cannot use the slack variables as initial basic variable anymore because they are not feasible (violate the non-negative constraint).

When it is difficult to find a basic feasible solution for the original problem, we create an artificial problem that we know is feasible. For the example, we add x_5 and get:

```
max - x_1 - 3x_2
s.t. x_1 + 2x_2 - x_3 + x_5 = 1
5x_1 + x_2 + x_4 = 10
x_1, x_2, x_3, x_4, x_5 \ge 0
```

 x_5 is called an "artificial variable" because it's neither a decision nor a slack variable. It is now easy to obtain initial basic variables (x_5 and x_4).

Problem: A feasible solution to this problem might not be feasible to the original problem. For example, we could have x_1, x_2, x_3 equals 0 and $x_5 = 1$. This violate the original constraint $x_1 + 2x_2 \ge 1$. This is caused by the *non-zero artificial variable* in the results!

Solution: Force the artificial variable (x_5) to be 0. There are two methods.

The problem can be solved in two steps. First we create another LP problem of minimizing x_5 subject to the same constraints. This is called the **Phase I** problem.

```
max \quad x_0 = -x_5
s.t. x_1 + 2x_2 - x_3 + x_5 = 1
5x_1 + x_2 + x_4 = 10
x_1, x_2, x_3, x_4, x_5 \ge 0
```

This can be solved using the Simplex method.

- If the optimal objective function value is 0, we have a feasible solution.
- The optimal point of the Phase I problem provides a set of initial basis for the original problem. We can then eliminate the artificial variables and use the optimal point of the Phase I problem as initial value to solve the original problem.

Put into tableau, get

	<i>x</i> ₀	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
<i>x</i> ₀	1	0	0	0	0	1	0
<i>x</i> ₅	0	1	2	-1	0	1	1
<i>x</i> ₄	0	5	1	0	1	0	10

First eliminate the coefficient for artificial variables: new Row 1=Old row 1 - Old row 2):

	<i>x</i> ₀	x_1	x_2	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
<i>x</i> ₀	1	-1	-2	1	0	0	-1
<i>x</i> ₅	0	1	2	-1	0	1	1
<i>x</i> ₄	0	5	1	0	1	0	10

Pivot out x_5 , and pivot in x_2 , get:

	<i>x</i> ₀	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	<i>x</i> ₅	RHS
<i>x</i> ₀	1	0	0	0	0	1	0
<i>x</i> ₂	0	1/2	1	-1/2	0	1/2	1/2
<i>x</i> ₄	0	9/2	0	1/2	1	-1/2	19/2

Now x_0 is optimized at $x_5 = 0$. We can eliminate x_5 and use $x_2 = 1/2$, $x_4 = 19/2$ as initial solution for the original problem (this is the **Phase II** problem). The tableau for the initial problem is:

	Z	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	RHS
Z.	1	1	3	0	0	0
<i>x</i> ₂	0	1/2	1	-1/2	0	1/2
<i>x</i> ₄	0	9/2	0	1/2	1	19/2

This is not a valid tableau!! (why?) Need to adjust and get:

	Z	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>x</i> ₄	RHS
Z	1	-1/2	0	3/2	0	-3/2
<i>x</i> ₂	0	1/2	1	-1/2	0	1/2
<i>x</i> ₄	0	9/2	0	1/2	1	19/2

Continue to finish!

Another technique is to modify the objective function to include artificial variable, but with a big penalty:

 $max - x_1 - 3x_2 - Mx_5$ s.t. $x_1 + 2x_2 - x_3 + x_5 = 1$ $5x_1 + x_2 + x_4 = 10$ $x_1, x_2, x_3, x_4, x_5 \ge 0$

M is assumed to be huge that it dominates the objective function.

- Using usual Simplex, all artificial variables will be first pivot out and become NBV because the using of M.
- Once the artificial variables are out, we obtain a set of initial basis for the original problem. We can then eliminate the artificial variables and solve the original problem the usual way.

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To solve a general LP with \leq , \geq or = constraints:

- 1. Make all right hand size ≥ 0 .
- 2. Add slack variables for \leq constraints, and surplus variable for \geq constraints.
- 3. Add artificial variables for \geq or = constraints.
- 4. Use slack and artificial variables as initial basic variables.
- 5. Set up Phase I problem or use big-M method to pivot out all artificial variables (e.g., make all artificial variables nonbasic variables).
- 6. Use the optimal solution from Phase I or big-M as initial solution, eliminate artificial variables from the problem and finish the original problem.

There are a large number of LP solver software both commercial or freely available. See Wikipedia page of "linear programming" for a list.

- In R, Simplex method is implemented as simplex function in boot package.
- In Matlab, the optimiztion toolbox contains linprog function.
- IBM ILOG CPLEX is commercial optimization package written in C. It is very powerful and highly efficient for solving large scale LP problems.

simplex

package:boot

R Documentation

Simplex Method for Linear Programming Problems

Description:

This function will optimize the linear function a%x subject to the constraints A1%x <= b1, A2%x ×> = b2, A3%x = b3 and x >= 0. Either maximization or minimization is possible but the default is minimization.

Usage:

simplex(a, A1 = NULL, b1 = NULL, A2 = NULL, b2 = NULL, A3 = NULL, b3 = NULL, maxi = FALSE, n.iter = n + 2 * m, eps = 1e-10)

```
> library(boot)
> a = c(50, 30, 40)
> A1 = matrix(c(2,3,5,5,2,4), nrow=2, byrow=TRUE)
> b1 = c(100, 80)
> simplex(-a, A1, b1)
```

Linear Programming Results

Call : simplex(a = -a, A1 = A1, b1 = b1)

Minimization Problem with Objective Function Coefficients x1 x2 x3 -50 -30 -40

Optimal solution has the following values x1 x2 x33.636364 30.909091 0.000000 The optimal value of the objective function is -1109.09090909091.

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For a typical maximizing LP problem like the following (with 3 variables and 2 constraints):

 $max \quad c_1 x_1 + c_2 x_2 + c_3 x_3$ s.t. $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \le b_1$ $a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \le b_2$ $x_1, x_2, x_3 \ge 0$

Economical interpretation of the problem:

- We produce three products, using two materials.
- x_j : unit of production of product j, j = 1, 2, 3. Unknown to be obtained.
- c_j : profit per unit of product j, j = 1, 2, 3.
- a_{ij} : unit of material i (i = 1, 2) required to produce 1 unit of product j.
- b_i : unit of available material i, i = 1, 2.

The goal is to maximize the profit, subject to the material constraints.

Now assume a buyer consider to buy our entire inventory of materials but not sure how to price the materials, but they know that we will only do the business if selling the materials yields higher return than producing the product.

Buyer's business strategy: producing one unit **less** of product *j* will save us:

• a_{1i} unit of material 1, and a_{2i} unit of material 2.

Buyer want to compute the unit prices of materials to **minimize** their own cost, subject to the constraints that we will do business (that we will not make less money). Assume the unit price for the materials are (unknown) y_1 and y_2 , the buyer will face the following optimization problem (called **Resource valuation problem**):

```
\begin{array}{ll} \min & b_1 y_1 + b_2 y_2 \\ s.t. & a_{11} y_1 + a_{21} y_2 \ge c_1 \\ & a_{12} y_1 + a_{22} y_2 \ge c_2 \\ & a_{13} y_1 + a_{23} y_2 \ge c_3 \\ & y_1, y_2 \ge 0 \end{array}
```

Duality

The buyer's LP problem is called the "**dual**" problem of the original problem, which is called the "**primal** problem".

In matrix notation, if the **primal** LP problem is:

 $\begin{array}{ll} max & cx \\ s.t. & Ax \leq b, x \geq 0 \end{array}$

The corresponding dual problem is:

 $\begin{array}{ll} \min \quad b^T y\\ s.t. \quad A^T y \geq c^T, y \geq 0 \end{array}$

Or to express in the canonical form (a maximization problem with \leq constraints):

$$max - b^{T}y$$

s.t. $-A^{T}y \le -c^{T}, y \ge 0$

Dual is the "negative transpose" of the primal. It's easy to see, the dual of the dual problem is the primal problem.

What if the primal problem doesn't fit into the canonical form (e.g., with \geq or = constraints, unrestricted variable, etc.)? The general rules of converting are:

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• The variable types of the dual problem is determined by the constraints types of the primal:

Primal (max) constraints	Dual (min) variable
\leq	≥ 0
\geq	≤ 0
=	unrestricted

 Conversely, the constraints types of the dual problem is determined by the variable types of the primal:

Primal (max) variable	Dual (min) constraints
≥ 0	≥
≤ 0	≤
unrestricted	=

If the primal problem is:

$$max \quad 20x_1 + 10x_2 + 50x_3$$

s.t.
$$3x_1 + x_2 + 9x_3 \le 10$$

$$7x_1 + 2x_2 + 3x_3 = 8$$

$$6x_1 + x_2 + 10x_3 \ge 1$$

$$x_1 \ge 0, x_2 \text{ unrestricted}, x_3 \le 1$$

0

The dual problem is:

$$\begin{array}{ll} \min & 10y_1 + 8y_2 + y_3 \\ s.t. & 3y_1 + 7y_2 + 6y_3 \geq 20 \\ & y_1 + 2y_2 + y_3 = 10 \\ & 9y_1 + 3y_2 + 10y_3 \leq 50 \\ & y_1 \geq 0, y_2 \ \textit{unrestricted}, y_3 \leq 0 \end{array}$$

Weak duality Theorem: the objective function value of the primal problem (max) at any feasible solution is always less than or equal to the objective function value of the dual problem (min) at any feasible solution.

So if (x_1, \ldots, x_n) is a feasible solution for the primal problem, and (y_1, \ldots, y_m) is a feasible solution for the dual problem, then $\sum_j c_j x_j \le \sum_i b_i y_i$.

Proof:

$$\sum_{j} c_{j} x_{j} \leq \sum_{j} \left(\sum_{i} y_{i} a_{ij} \right) x_{j}$$
$$= \sum_{i,j} y_{i} a_{ij} x_{j}$$
$$= \sum_{i} \left(\sum_{j} a_{ij} x_{j} \right) y_{i}$$
$$\leq \sum_{i} b_{i} y_{i}.$$

In other words, we will only do business if selling the material makes us more money.

- We know that at feasible solutions for both, the objective function of the dual problem is always greater or equal.
- A question is that if there is a difference between the largest primal value and the smallest dual value? Such difference is called the "**Duality gap**".
- The answer is provided by the following theorem.

Strong Duality Theorem: If the primal problem has an optimal solution, then the dual also has an optimal solution and there is no duality gap.

Economic interpretation: at optimality, the resource allocation and resource valuation problems give the same objective function values. In other words, in the ideal economic situation, using the materials or selling the materials give the same profit.

Recall for the LP problem in standard form: $max \ z = cx, s.t. \ Ax \le b, x \ge 0$. Let x^* be an optimal solution. Let *B* be the columns of *A* associated with the BV, and *R* is the set of columns associated with the NBV. We have $c_B B^{-1} a_j - c_j \ge 0$ for $\forall j \in R$.

- Define $y^* \equiv (c_B B^{-1})^T$, we will have $y^{*T} A \ge c$.
- Furthermore, $y^{*T} \ge 0$ (why?).

Thus y^* is a feasible solution for the dual problem.

How about optimality? We know $y^{*T}b \ge y^{*T}Ax \ge cx$. Want to prove $y^{*T}b = cx^*$. This requires two steps:

1. $y^{*T}b = y^{*T}Ax^* \Leftrightarrow y^{*T}(b - Ax^*) = 0$

2. $y^*Ax^* = cx^* \Leftrightarrow (y^{*T}A - c)x^* = 0$

This can be seen from the optimal tableau:

	Z	x_B	x_N	RHS
Z	1	0	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$
x_B	0		$B^{-1}N$	$B^{-1}b$

An optimal solution $x^* = [x_B^*, 0]^T$, A = [B, N], $c = [c_B, c_N]$:

1. becomes $c_B B^{-1} b - c_B B^{-1} [B, N] [x_B^*, 0]^T = c_B B^{-1} b - c_B x_B^* = 0$

2. becomes $(c_B B^{-1}[B, N] - [c_B, c_N])[x_B^*, 0]^T = [0, c_B B^{-1} N - c_N][x_B^*, 0]^T = 0$

Review

• Linear programming (LP) is an constrained optimization method, where both the objective function and constraints must be linear.

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- If an LP has an optimal solution, it must have an extreme point optimal solution. This great reduces the search space.
- Simplex is a method to search for an optimal solution. It goes through the extreme points and guarantees the increase of objective function at every step. It has better performance than exhaustive search the extreme points.
- Primal-dual property of the LP.
- Weak and strong duality.