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Reviewed work(s):

Source: *The American Mathematical Monthly*, Vol. 76, No. 8 (Oct., 1969), pp. 916-918

Published by: [Mathematical Association of America](#)

Stable URL: <http://www.jstor.org/stable/2317945>

Accessed: 25/08/2012 17:04

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MATHEMATICAL NOTES

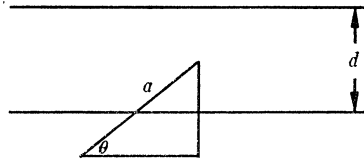
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BUFFON'S NOODLE PROBLEM

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In 1733, Georges Louis Leclerc, Comte de Buffon, considered the following problem: Given a needle of length a and an infinite grid of parallel lines with common distance d between them, what is the probability $P(E)$ that a needle, tossed at the grid randomly, will cross one of the parallel lines?



If $a \leq d$ and we choose a sector $\Delta\theta$ as the direction in which the needle is to fall, we see that $P(\theta \in \Delta\theta) = (\Delta\theta)/\pi$ and $P(\text{cross} \mid \theta \in \Delta\theta) = (a \sin \theta)/d$. Thus $P(E)$ may be calculated using the formula for conditional probability $P(AH) = P(A \mid H)P(H)$ and integrating θ from 0 to π as follows:

$$P(E) = \int_0^\pi \frac{a \sin \theta d\theta}{\pi d} = (a/\pi d) \int_0^\pi \sin \theta d\theta = 2a/\pi d.$$

If $d < a$, the calculations are rougher, but

$$P(E) = (2a/\pi d)(1 - \cos \alpha) + (\pi - 2\alpha)/\pi \quad \text{where} \quad \alpha = \arcsin(d/a),$$

which actually coincides with the earlier result if $d \geq a$.

Gnedenko [1, p. 43] generalized the problem first to n -sided convex polygons with diameter less than d , and then to convex closed curves with diameter less than d by considering such curves as limits of inscribed polygons, giving the formula $P(\text{cross}) = a/\pi d$. The requirement on the diameter was given to insure a probability $P \leq 1$, for in the case of a circle with diameter greater than d , the probability of a cross is always 1 although the formula gives a value greater than 1. The convexity requirement on the polygons allowed one to assert that, with probability one, the polygon crosses a line if and only if exactly two sides of the polygon cross the line. A computational proof was then given to find the probability of this latter event.

In order to drop these assumptions on the curve (closed, convex, and with restricted diameter), we generalize the problem in another direction by asking, "How many lines might we expect the needle to cross?" To do this we shall say that there are exactly n *line-crossings* (or just *crossings*) on a given toss if there

are exactly n points of intersection of the needle with the infinite grid. In the case of a straight needle N of length a , the expected number of line-crossings $e(N)$ may be simply calculated as $e(N) = 2a/\pi d$. We shall obtain this as a corollary of the following result showing that e is a function of the length of the needle only, not its shape.

THEOREM. *Let N be a wet noodle of length a thrown at random onto an infinite grid of parallel lines with common distance d between them. Then the expected number of line-crossings $e(N)$ is given by $e(N) = 2a/\pi d$.*

Before giving the proof we remark that our definition of line-crossing demands that we count as two distinct crossings the event that the noodle crosses the same line in two places. Also we are, of course, ignoring the physical fact that the number of crossings must be an integer, and we are looking at the problem as one asking for expectation. Finally, the result may seem more reasonable if we think of bending a needle of length $\leq d$ in half. The probability that the bent needle crosses a line will indeed be halved, but the expected number of crossings will remain the same since each cross will result in two *crossings* (one for each half of the bent needle).

Proof. We choose a sequence of polygonal lines $L_1, L_2, \dots$ that approaches the curve N uniformly (this we may do since N has a length and is thus rectifiable). The segments of the line L_i we shall denote as $L_{i1}, L_{i2}, \dots, L_{in}$, and we shall write $L_i = L_{i1} + L_{i2} + \dots + L_{in}$. Letting a_{ij} denote the length of L_{ij} , we may suppose n sufficiently large to insure that $a_{ij} < d$ for all i, j so that $e(L_{ij})$ is just the probability that L_{ij} crosses a line, viz., $e(L_{ij}) = 2a_{ij}/\pi d$. Now letting a_i be the length of L_i , we know that a_i approaches a as i tends to infinity. Hence, if $e(L_i) = e(L_{i1}) + \dots + e(L_{in})$ for all i , then

$$e(L_i) = \sum_j (2a_{ij}/\pi d) = (2/\pi d) \sum_j (a_{ij}) = (2/\pi d) a_i$$

and in the limit $e(N) = 2a/\pi d$. It remains then to show that $e(L_i) = e(L_{i1}) + \dots + e(L_{in})$ for all i . This follows immediately by induction on n once the case $n = 2$ is established. So suppose L and L' to be segments of a polygonal path $L + L'$ with both L and L' of length less than d . We consider the three events:

- A. Only L crosses a line,
- B. Only L' crosses a line,
- C. Both L and L' cross a line (or lines).

The restriction on the lengths of L and L' insure that an occurrence of either A or B will result in exactly one line-crossing of $L + L'$, while an occurrence of C will result in exactly two line-crossings of $L + L'$. Thus

$$e(L + L') = e(A) + e(B) + 2e(C).$$

But since $e(L) = e(A) + e(C)$ and $e(L') = e(B) + e(C)$, we have the result

$$e(L + L') = e(L) + e(L').$$

Gnedenko's result then follows since, under his hypothesis, the event E that the curve intersects a line of the grid would occur, with probability one, if and only if there occur exactly two line-crossings. Hence $P(E)$ is exactly one-half the expected number of line-crossings, $P(E) = (1/2)(2a/\pi d) = a/\pi d$, where a is the perimeter of the curve.

We remark that this result may be obtained geometrically by taking the "grid's point of view" and considering the experiment as one of tossing the grid onto a noodle fixed in the plane. The expected number of crosses is independent of the position of the noodle, and in fact one could change the noodle's position after each throw.

Reference

1. B. V. Gnedenko, *The Theory of Probability*, 2nd ed., Chelsea, New York, 1963.

AN APPLICATION OF TWO ESTIMATES FOR e

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A. F. Sololev recently obtained the following formula for the jump at a point $a \in (0, 1)$ of a nondecreasing function g defined on $[0, 1]$:

$$(S) \quad g(a^+) - g(a^-) = \inf_{n=1,2,\dots} \inf_{k=1,\dots,n} \int_0^1 \frac{x^k(1-x)^{n-k}}{a^k(1-a)^{n-k}} dg.$$

Sololev came to g via a Hausdorff moment problem and obtained his formula in that setting. The purpose of this note is to give an elementary verification of his formula based on two estimates for e which arise in a beginning calculus class:

$$u(x) = \left(1 + \frac{1}{x}\right)^x \text{ increases to } e$$

and

$$v(x) = \left(1 + \frac{1}{x}\right)^{x+1} \text{ decreases to } e$$

as x increases from 1.

Because elementary properties of the Riemann-Stieltjes integral tell us that the right side of (S) is greater than or equal to the left, it is the reverse inequality to which we shall direct our attention. Let f be defined on $[0, 1]$ by $f(x) = x^k(1-x)^{n-k}$, where n and k remain to be chosen. Then f obtains its max at k/n , and a simple computation gives

$$\frac{f\{(k-1)/n\}}{f(k/n)} = \frac{u(n-k)}{v(k-1)} \quad \text{and} \quad \frac{f\{(k+1)/n\}}{f(k/n)} = \frac{u(k)}{v(n-k-1)}.$$

Moreover, if each of n , k , and $n-k$ is large and each of a and $x \in [(k-1)/n,$